4. Banch Spaces.

A Banach Space is a vector space $\mathbf{X}$ with a norm $\|x\|$ satisfying $\|cx\| = |c|\|x\|$ for real or complex scalars c and $\|x+y\| \le \|x\| + \|y\|$. In addition $\mathbf{X}$ is complete under the metric $d(x,y) = \|x-y\|$.

Examples. L_p, ℓ_p for $1 \le p \le \infty$. $C(X)$ the space of continuous functions on a compact space X, or $C_b(X)$ the space of bounded continuous functions on X. The norms being $[\int |f(x)|^p d\mu]^{\frac{1}{p}}$ or $[\sum_{n=0}^{\infty} |a_n|^p]^{\frac{1}{p}}$ for $1 \le p < \infty$. On L_∞ it is the essential supremum and on ℓ_∞ it is $\sup_n |a_n|$ with $\|f\| = \sup_x |f(x)|$ on the space of bounded continuous functions.

Linear Functions. They are linear maps $\Lambda : \mathbf{X} \to R$ or $\mathbf{X} \to C$. Bounded or continuous linear functionals are those that satisfy $|\Lambda(x)| \le C\|x\|$. With $\|\Lambda\| = \sup_{\|x\|\le 1} |\Lambda(x)$ as norm the set of linear functionals is a Banach space called the dual $\mathbf{X}^*$. For $p > 1$ the dual of L_p is L_q where $q = \frac{p}{p-1}$ with $q = \infty$ when $p = 1$. But the dual of L_∞ is bigger than L_1. The second dual is $[\mathbf{X}^*]^*$ and contains $\mathbf{X}$. but could be bigger. If it is the same $\mathbf{X}$ is said to be reflexive. For $1 < p < \infty$, L_p is reflexive while L_1 and ℓ_1 are not unless they are finite dimensional.

Linear Operators. They are linear maps $\{T : \mathbf{X} \to \mathbf{Y}\}$ that are continuous or bounded if $\|Tx\| \le C\|x\|$ and such operators form a Banach space with norm $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$. If $\{T_1 : \mathbf{X} \to \mathbf{Y}\}$ and $\{T_2 : \mathbf{Y} \to \mathbf{Z}\}$ then $\{T_2T_1 : \mathbf{X} \to \mathbf{Z}\}$ with $\|T_2T_1\| \le \|T_2\|\|T_1\|$. If $\{T : \mathbf{X} \to \mathbf{Y}\}$ then $\{T^* : \mathbf{Y}^* \to \mathbf{X}^*\}$ and $\|T^*\| = \|T\|$. $(T_2T_1)^* = T_1^*T_2^*$.

Baire category theorem. If X is complete metric space and $X = \cup_{j=1}^{\infty} C_j$ is a countable union of closed sets, then at least one C_j must have a nonempty interior, i.e. C_j contains an open ball $S(x,\epsilon)$ around some point for some $\epsilon > 0$.

Proof. Let C_1, C_2 be two closed sets such that their union $C_1 \cup C_2$ has a nonempty interior. Then at least one of them must have an interior. To see this, let $x \in C_1$ and $S(x,\delta)$ be not a subset of $C_1 \cup C_2$. There is then $x' \in S(x,\delta) \cap C_1^c$ and consequently $S(x',\delta') \subset C_1^c$ for some $\delta' > 0$. Since $S(x',\delta') \subset (C_1 \cup C_2) \cap C_1^c$ it must be contained in C_2.

Let $X = \cup_{j=1}^{\infty} C_j$. If $X = \cup_{j=1}^{n} C_j$ for some finite n we are done (by induction on n). We can find nested balls $S(x_j, \delta_j) \downarrow$ with $\delta_j \to 0$. x_j is a Cauchy sequence with a limit $x \in \cap_j S(x_j, \delta_j)$. $x \notin \cup_{j=1}^{n} C_j$ implying $x \notin X = \cup_{j=1}^{\infty} C_j$.

Open mapping theorem. Let T be a bounded map fro $\mathbf{X}$ *onto* $\mathbf{Y}$. Then the image TS of the unit ball S in $\mathbf{X}$ has nonempty interior. Equivalently the image of any open set is open. Or the image of the init ball $\{x : \|x\| \le 1\}$ in X contains a ball $\{y : \|y\| \le \delta\}$ of some positive radius in $\mathbf{Y}$. If T is a bounded one to one and onto map from $\mathbf{X}$ onto $\mathbf{Y}$, then T^{-1} is bounded.

Proof. Since T is onto $\cup_{k=1}^{\infty} TB(0;k) = \mathbf{Y}$. By Baire category theorem for some k_0, $\overline{TB(0,k_0)}$ then contains an open set $B(y_0,\delta)$ around some x_0. Set of points of the form $T(x_1 - x_2)$ with x_1, x_2 from $S(0,k_0)$ will then contain a Ball of radius 2δ around 0. Any point $y \in \mathbf{Y}$ with $\|y\| \le 2\delta$ is arbitrarily close to Tx for some x in $B(0,2k_0)$. By scaling any point in $B(0,a)$ in Y is arbitrarily close to a point in the image of $B(0,\theta a)$ where $\theta = k_0\delta^{-1}$. Let $y \in B(0,1) \subset \mathbf{Y}$. Find $x_1 \in B(0,\theta)$ such that $\|y - Tx_1\| < \frac{1}{2}$, Then if

$y_1 = y - Tx_1$, $\|y_1\| \le \frac{1}{2}$. We can find x_2 with $\|x_2\| \le \frac{\theta}{2}$ such that $\|y_1 - Tx_2\| = \|y_2\| \le \frac{1}{4}$. Proceeding we have

$$y = Tx_1 + Tx_2 + \cdots + Tx_n + y_n$$

with $\|x_n\| \le \theta 2^{-(n-1)}$. Now $x = \sum_n x_n$ exists $\|x\| \le 2\theta$ and $Tx = y$. The map T is open.

Uniform Boundedness Principle. Let $\{T_\alpha\}$ are bounded linear maps from Banach space $\mathbf{X}$ to Banach space $\mathbf{Y}$ such that $\sup_\alpha \|T_\alpha x\| = C(x) < \infty$ for every $x \in \mathbf{X}$. Then $C(x) \le C\|x\|$ for some constant C, i.e. $\sup_\alpha \|T_\alpha\| < \infty$

Proof. Let $C_n = \{x : C(x) \le n\}$. C_n is closed and $\cup_n C_n = \mathbf{X}$. Some C_n has interior. There is a an open ball $S(x_0, \delta)$ contained in some C_k and C_{2k} will contain $S(0, 2\delta)$. Since $C(rx) = rC(x)$ for $r > 0$, it follows that $C(x) \le \frac{k}{\delta}\|x\|$.

Closed Graph Theorem. If T maps $\mathbf{X} \to \mathbf{Y}$ the graph of T is the linear set of points $(x, Tx) \in \mathbf{X} \times \mathbf{Y}$ as x varies over $\mathbf{X}$. The closed graph theorem says that if the graph of T is a closed subspace of $\mathbf{X} \times \mathbf{Y}$ then T is necessarily bonded.

Proof. Let $\mathbf{Z} = \mathbf{X} \oplus \mathbf{Y}$ and $M = \{(x, Tx)\}$ the graph of T is a closed subspace of $\mathbf{Z}$. Then $\mathbf{X}$ can have a new norm $\|x\| + \|Tx\|$ under which it is again a Banach space. To check completeness means proving that if x_n and Tx_n are both Cauchy then the limit is (x, y) with $y = Tx$. This is precisely the graph being closed in $\mathbf{X} \oplus \mathbf{Y}$. The map $(x, Tx) \to x$ is clearly, bounded, one to one and onto. The inverse $x \to (x, Tx)$ is also then bounded.

Hahn-Banach Theorem. Given a linear functional $\Lambda(x)$ from a closed subspace $\mathbf{Y} \subset \mathbf{X}$ satisfying $|\Lambda(x)| \le p(x)$ where p, defined on $\mathbf{X}$, satisfies $p \ge 0, p(ax) = |a|p(x)$ and $p(x+y) \le p(x) + p(y)$, p can extended from $\mathbf{Y}$ to $\mathbf{X}$ satisfying $|\Lambda(x)| \le p(x)$ for all $x \in X$.

Proof. Take $x_0 \notin \mathbf{Y}$. Let us define $\Lambda(x + cx_0) = \Lambda(x) + ca$ for some $a \in R$. Need to pick a such that $\Lambda(x) + ca \le p(x + cx_0)$ for all $x \in \mathbf{Y}$ and $c \in R$.

$$\sup_{c \ge 0} \frac{\Lambda(x) - p(x - cx_0)}{c} \le a \le \inf_{c > 0} \frac{p(x + cx_0) - \Lambda(x)}{c}$$

For this to be possible we need for $c_1, c_2 > 0, x \in \mathbf{Y}$,

$$\frac{\Lambda(x) - p(x - c_1 x_0)}{c_1} \le \frac{p(x + c_2 x_0) - \Lambda(x)}{c_2}$$

or

$$c_2[\Lambda(x) - p(x - c_1 x_0)] \le c_1[p(x + c_2 x_0) - \Lambda(x)]$$

$$\Lambda(x) \le \frac{c_1 p(x + c_2 x_0) + c_2(x - c_1 x_0)}{c_1 + c_2}$$

follows from sub-additivity and homogeneity of p.

$$\begin{aligned} p(x) &= p(\frac{c_1}{c_1 + c_2}x + \frac{c_1 c_2}{c_1 + c_2}x_0 + \frac{c_2}{c_1 + c_2}x - \frac{c_1 c_2}{c_1 + c_2}x_0) \\ &\le p(\frac{c_1}{c_1 + c_2}x + \frac{c_1 c_2}{c_1 + c_2}x_0) + p(\frac{c_2}{c_1 + c_2}x - \frac{c_1 c_2}{c_1 + c_2}x_0) \\ &= \frac{c_1}{c_1 + c_2}p(x + c_2 x_0) + \frac{c_2}{c_1 + c_2}p(x - c_2 x_0) \end{aligned}$$

Problem 4.1 Let $x_1, \ldots, x_d$ be d linearly independent vectors in a Banach space X and V their linear span. Show that V is a closed subspace of X and there exists a complementary closed subspace $Y \subset X$ such that $X = Y \oplus V$. In any other decomposition of $X = Y \oplus W$ the dimension of W must be d.

A subspace M (not assumed to be closed) is of finite co-dimension d in a Banach space X if it is spanned by M and a finite number d of lvectors $x_1, \ldots, x_d$ that are linearly independent modulo M.

Theorem. A subspace of finite co-dimension is necessarily closed and the co-dimension d is well defined. There is a complementary subspace V of dimension d such that $X = M \oplus V$.

Proof. The quotient space $\mathbf{X}/M = V$ is a vector space and its dimension d is well defined. Any $x \in \mathbf{X}$ can be written as a unique sum $x = y + \sum_{i=1}^{k} \Lambda_i(x) x_i$ with $y \in M$. $\mathbf{X} = M \oplus V$ with V being the span of $\{x_1, \ldots, x_d\}$. The graph of the map $x \to \{\Lambda_i(x)\}$ of $X \to R^d$ is closed. It is then bounded and $M = \cap_{i=1}^{k} \{x : \Lambda_i(x) = 0\}$ is closed.